THE GEOMETRY OF RELATIONS

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Abstract. The classical way to study a finite poset $(X, \leq)$ using topology is by means of the simplicial complex Δ_X of its nonempty chains. There is also an alternative approach, regarding X as a finite topological space. In this article we introduce new constructions for studying X topologically: inspired by a classical paper of C.H. Dowker [9], we define the simplicial complexes K_X and L_X associated to the relation $\leq$. In many cases these polyhedra have the same homotopy type as the order complex Δ_X. We give a complete characterization of the simplicial complexes that are the K or L-complexes of some finite poset and prove that K_X and L_X are topologically equivalent to the smaller complexes K'_X, L'_X induced by the relation $<$. More precisely, we prove that K_X (resp. L_X) simplicially collapses to K'_X (resp. L'_X). The paper concludes with a result that relates the K-complexes of two posets X, Y with closed relations $R \subset X \times Y$.

1. Introduction

There is a construction, introduced by C.H. Dowker in [9], that associates two simplicial complexes K and L to a relation from one set to another. Dowker proved that the polyhedra $|K|$ and $|L|$ are homotopy equivalent and applied this result to relate the Čech and Vietoris homology groups of a topological space. Other interesting applications of Dowker's result are given in [2] and [8]. In those papers these constructions are used to study the homotopy theory of *global actions* and *groupoid atlases.* In [2, Prop.7.7] it is proved that the fundamental group of a groupoid atlas coincides with the fundamental group of the nerve of its cover by local components. In [8], the *strong fundamental group* and the homology groups of groupoid atlases are defined in terms of nerves.

In this article we investigate the topology of finite posets from an alternative point of view, based on Dowker's construction of the K and L complexes associated to a relation. The classical way to study a finite poset $(X, \leq)$ using topology is by means of its order complex Δ_X. This complex has been widely investigated by many authors and has implications in combinatorics, algebraic topology and combinatorial geometry (see [6] and [10] for standard applications of this complex). For example, in [16] Quillen analyzed the homotopy properties of the poset of p-subgroups of a finite group in terms of the order complex. There is an alternative approach to study finite posets topologically, which goes back to Alexandroff [1], regarding the poset as a finite T_0-space (see Section 2 for more details). By a theorem of McCord [12], the order complex of a finite poset and its corresponding finite T_0-space are weak homotopy equivalent; in particular, they have the same homology and homotopy groups. Finite spaces have been recently investigated in a series of joint papers with J. Barmak [3, 4, 5]. There is also a beautiful survey by P. May [11]. Here we introduce, for a finite poset $(X, \leq)$, the simplicial complexes K_X and

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L_X associated to the relation $R \subset X \times X$ given by $\leq$ and compare the geometry of these complexes with the geometry of the classical order complex. Under certain hypotheses on the poset, these complexes are homotopy equivalent to the order complex.

We also introduce the smaller complexes K'_X and L'_X associated to the relation $<$ of the poset. We show that K_X simplicially collapses to K'_X and L_X collapses to L'_X. This result allows to study topological properties of a finite poset $(X, \leq)$ using the simpler complexes K'_X and L'_X, which have the same simple homotopy type as the complexes K_X and L_X.

The paper ends with a result that relates the K-complexes of two posets X, Y with *closed* relations $R \subset X \times Y$. This result is a variation of a Theorem by Quillen which relates the order complexes of the posets [16].

The rest of the paper is organized as follows. In Section 2 we recall Dowker's construction and the correspondence between finite posets and finite spaces. In Section 3 we prove a Galois-type correspondence between equivalence classes of relations $R \subset X \times Y$ defined on a fixed nonempty set X and the subcomplexes of the simplex spanned by the elements of X. In Section 4 we introduce and investigate the complexes K_X and L_X associated to a finite poset $(X, \leq)$. We give a complete characterization of the complexes which are the K or L-complexes of some finite poset and compare the topology of K_X (resp. L_X) with the topology of K'_X (resp. L'_X). Section 5 is devoted to the study of the relationship between the K-complexes of two posets with a closed relation.

2. Preliminaries

In this section we recall Dowker's construction of the $K-$ and L-complexes and the relationship between finite spaces, finite posets and simplicial complexes.

2.1. **Dowker's construction.** A relation between two nonempty sets X and Y is a subset R of the cartesian product $X \times Y$. We will write xRy if $(x, y) \in R$. There is a canonical way to associate to R two simplicial complexes K and L.

Definition 2.1. The simplicial complex K is defined as follows. The n-simplices of K are the finite subsets $\{x_0, \ldots, x_n\}$ of X such that there exists some $y \in Y$ with $x_i R y$ for all $i = 0, \ldots, n$. Similary, the simplices of L are the finite subsets σ of Y such that the elements of σ are related to a common element of X. In particular, the set of vertices of K is a subset of X and consists of the points of X which are related to some element in Y. Analogously, the set of vertices of L is a subset of Y. We refer to K and L as the K-complex and the L-complex associated to the relation R.

These constructions were introduced by C.H. Dowker in [9] where he proved the following result.

Theorem 2.2 (C.H. Dowker)**.** *Let $R \subset X \times Y$ be a relation and let K and L be the associated simplicial complexes. Then the polyhedra $|K|$ and $|L|$ are homotopy equivalent.*

Here $|K|$ denotes the geometric realization of the simplicial complex K.

Dowker applied this result to relate the Čech and Vietoris homology groups of a space. More precisely, let X be a topological space and $\mathcal{U}$ a cover of X by subsets, i.e. $X = \bigcup_{U \in \mathcal{U}} U$. Consider the relation xRU if $x \in U$. The K-complex is called in this case the Vietoris complex of the covering, and it is denoted by $V(\mathcal{U})$, and the L-complex is called the nerve of the covering, which is denoted by $\mathcal{N}(\mathcal{U})$. Since the geometric realizations of $V(\mathcal{U})$ and $\mathcal{N}(\mathcal{U})$ are homotopy equivalent, it follows that the Čech and Vietoris homology coincide.

2.2. **Finite posets and finite spaces.** In this article we investigate finite posets from a geometrical viewpoint using the $K-$ and L-complexes associated to the relations $\leq$ and $<$ of the posets.

Sometimes it is very useful to regard finite posets as finite topological T_0-spaces. Recall that a topological space X is said to be T_0 if for every pair of points of X there exists some open set containing one and only one of those points. The correspondence between finite T_0-spaces and finite posets is as follows. Given a finite poset $(X, \leq)$, we define for each $x \in X$ the set

$$U_x = \{y \in X \mid y \leq x\}.$$

It is not difficult to verify that these sets form a basis for a topology. This is the topology associated to $\leq$. Conversely, given a finite T_0 topological space X, we consider for each $x \in X$ the *minimal open set* U_x which is defined as the intersection of all open sets containing x. The partial order associated to the topology on X is given by the relation $x \leq y$ if $x \in U_y$. These applications define a one-to-one correspondence between T_0-topologies and partial orders on the finite set X. Therefore one can consider finite T_0-spaces as finite posets and viceversa. Order preserving functions correspond to continuous maps and lower sets to open sets. For more details on finite spaces, we refer to reader to the foundational articles [12, 21], P.May's notes [11] and the more recent articles [3, 4, 5]. See also [20] for the combinatorics of posets and [6, 10] for the topology of posets.

There classical way to study a finite poset $(X, \leq)$ topologically is by means of its *order complex* Δ_X. Recall that the simplices of Δ_X are the nonempty (finite) chains in X. This construction goes back to [1], it is closely related to Segal's construction of a *classifying space* of a category [17] and it was developed and used by many authors. We refer the reader to [6] and [10] for standard applications and results about Δ_X. The articles [12, 4, 5] relate the topology of a finite poset $(X, \leq)$ viewed as a finite topological space with the topology of the associated simplicial complex Δ_X. In this paper, the relationship between the intrinsic topology of a finite poset and the topology of the associated order complex Δ_X will not be explicitly used.

We will compare the classical complex Δ_X of a finite poset $(X, \leq)$ with new constructions which are based on the $K-$ and L-complexes associated to the relations $\leq$ and $<$ of the poset.

3. A Galois-type correspondence for relations

In this section we investigate the relationship between the different relations $R \subset X \times Y$ defined on a fixed nonempty set X and prove a Galois-type correspondence between the subcomplexes of the simplicial complex spanned by the set X and the equivalence classes of relations $R \subset X \times Y$.

Definition 3.1. Let X be a fixed nonempty set, not necessarily finite. Since X is fixed, a relation $R \subset X \times Y$ will be denoted by (Y, R). Let K_R be the K-complex and L_R the L-complex associated to (Y, R). A relation (Y, R) is called *covered* if the projection on Y, $p : R \to Y$, is onto or equivalently, if for any $y \in Y$ there is an element $x \in X$ such that xRy.

Let X be a fixed nonempty set. For the rest of this section, all relations (Y, R) are assumed to be covered.

Notation 3.2. We denote by K_* the simplicial complex spanned by the set X, i.e. the simplices of K_* are all finite subsets of X. When T is a subcomplex of some simplicial complex S, we will write $T \leq S$.

Note that for any (Y, R), $K_R \leq K_*$. Conversely, we have

Proposition 3.3. *For any $T \leq K_*$, there exists a (covered) relation (Y, R) such that $K_R = T$.*

Proof. Take $Y = S_T$, the set of simplices of T, and define $xR\sigma$ if $x \in \sigma$. It is easy to verify that $K_R = T$ □

Definition 3.4. A morphism $f : (Y, R) \to (Z, \tilde{R})$ is a set theoretic map $f : Y \to Z$ such that for every $x \in X$ and $y \in Y$,

$$xRy \Rightarrow x\tilde{R}f(y).$$

Remark 3.5. Note that a morphism $f : (Y, R) \to (Z, \tilde{R})$ induces a well defined simplicial map $L_f : L_R \to L_{\tilde{R}}$. Moreover, if there exists a morphism $f : (Y, R) \to (Z, \tilde{R})$, then K_R is a subcomplex of $K_{\tilde{R}}$.

Recall that two simplicial morphisms $\phi, \psi : N \to M$ are called contiguous if $\phi(\sigma) \cup \psi(\sigma)$ is a simplex of M for every simplex $\sigma \in N$ (cf. [19, Section 3.5]).

Remark 3.6. Let $f, g : (Y, R) \to (Z, \tilde{R})$ be morphisms. Then the induced simplicial morphisms $L_f, L_g : L_R \to L_{\tilde{R}}$ are contiguous. In particular, $|L_f|$ and $|L_g|$ are homotopic continuous maps.

Definition 3.7. Two relations (Y, R) and $(Z, \tilde{R})$ are called *equivalent* if there are morphisms $f : (Y, R) \to (Z, \tilde{R})$ and $h : (Z, \tilde{R}) \to (Y, R)$. We denote in this case $(Y, R) \simeq (Z, \tilde{R})$.

Note that $\simeq$ is an equivalence relation in the class of relations defined on X. To prove that it is transitive, note that any composition of morphisms is again a morphism. From remarks 3.5 and 3.6, one deduces the following

Corollary 3.8. *If $(Y, R) \simeq (Z, \tilde{R})$, then*

(a) $K_R = K_{\tilde{R}}$.
(b) *$|L_R|$ and $|L_{\tilde{R}}|$ are homotopy equivalent.*
(c) *Any morphism $g : (Y, R) \to (Z, \tilde{R})$ induces a homotopy equivalence $|L_g|$.*

Note that, if the hypothesis $(Y, R) \simeq (Z, \tilde{R})$ is not satisfied, a morphism $g : (Y, R) \to (Z, \tilde{R})$ does not in general induce a homotopy equivalence. Consider for example the constant morphism $(Y, R) \to (*, \tilde{R})$ where $*$ is the singleton and $x\tilde{R}*$ for all $x \in X$.

Note also that if $g : (Y, R) \to (Z, \tilde{R})$ induces a homotopy equivalence, this does not imply that (Y, R) and $(Z, \tilde{R})$ are equivalent. Consider for instance a relation (Y, R) such that L_R is contractible. In that case, the map $(Y, R) \to (*, \tilde{R})$ induces a homotopy equivalence but in general (Y, R) is not equivalent to $(*, \tilde{R})$ unless there exists $y \in Y$ such that xRy for all $x \in X$.

Definition and Remark 3.9. Given a relation (Y, R) and an element $y \in Y$, let S_y be the set of all elements of X which are related to y. Since we work with covered relations, these sets are nonempty. Moreover S_y is a generalized simplex of K_R, i.e. all its finite subsets are simplices of K_R. If S_y is finite, it is just a simplex in K_R.

We prove now the Galois-type correspondence for relations:

Theorem 3.10. *Let X be a finite set and let K_* be the simplicial complex spanned by X. There exists a one-to-one correspondence between subcomplexes of K_* and equivalence classes of covered relations (Y, R) on X. This correspondence assigns to each subcomplex T of K_* the class of the relation (S_T, R) as in proposition 3.3 and to each class (Y, R) the subcomplex $K_R \leq K_*$. Moreover,*

$$K_R \leq K_{\tilde{R}} \Longleftrightarrow \exists\ f : (Y, R) \to (Z, \tilde{R})$$

Proof. The first part of the Theorem follows from results 3.3, 3.5 and 3.8 of above.

Suppose now that (Y, R) and $(Z, \tilde{R})$ are relations on X with $K_R \leq K_{\tilde{R}}$. Let $y \in Y$ and let S_y be its associated generalized simplex. Since X is finite and $K_R \leq K_{\tilde{R}}$, then S_y is actually a simplex of $K_{\tilde{R}}$ and therefore there exists some $z \in Z$ such that $x\tilde{R}z$ for all $x \in S_y$. Define $f(y) = z$. The function $f : Y \to Z$ defined this way is a morphism of relations since

$$xRy \Longrightarrow x \in S_y \Longrightarrow x\tilde{R}f(y).$$

This completes the proof. □

Remark 3.11. Note that the finiteness hypothesis is needed to prove the implication

$$K_R \leq K_{\tilde{R}} \Longrightarrow \exists\ f : (Y, R) \to (Z, \tilde{R}).$$

If X is infinite, take $Y = P(X)$ the set of all nonempy subsets of X and let $Z = P_f(X)$ the set of all nonempty finite subsets of X and define xRT (and $x\tilde{R}T$) if $x \in T$. Then $K_R = K_{\tilde{R}}$ but there is no morphism $f : (Y, R) \to (Z, \tilde{R})$.

Example-Application 3.12. Let X be a topological space and let $\mathcal{U}$ and $\mathcal{V}$ be two covers of X by subsets, such that $\mathcal{V}$ refines $\mathcal{U}$, i.e. for any $V \in \mathcal{V}$ there is some $U \in \mathcal{U}$ such that $V \subseteq U$. If we consider $\mathcal{U}$ and $\mathcal{V}$ as relations on X as above, then there is a morphism of relations $f : \mathcal{V} \to \mathcal{U}$. In fact such a morphism is precisely a *refinement map* (or a *canonical projection* in the terminology of [19]). Therefore $\mathcal{V}$ refines $\mathcal{U}$ if and only if there is a morphism of relations $f : \mathcal{V} \to \mathcal{U}$ and two coverings refine each other if and only if they are equivalent (as relations on X). From Corollary 3.8 it follows the well-known fact that the nerves of two covers which refine each other are homotopy equivalent.

4. The K and L-complexes of a finite poset

Definition 4.1. Let $(X, \leq)$ be a finite poset. We denote by K_X and L_X the complexes associated to the relation $R \subset X \times X$ given by $\leq$. Note that the simplices of K_X are the subsets $\{x_0, \ldots, x_n\}$ of X such that there exists $y \in X$ with $x_i \leq y$ for all i. Similarly, the simplices of L_X are the subsets with a common lower bound $z \in X$.

Let $(X, \leq)$ be a finite poset, viewed as a finite T_0-space. Recall that the minimal open set $U_x = \{y \in X,\ y \leq x\}$ of an element x corresponds to the intersection of all the open subsets which contain x. Similarly, the minimal closed sets of X are the subspaces of the form $F_x = \{y \in X,\ x \leq y\}$, where F_x corresponds to the intersection of all closed subsets of X containing x. In view of Dowker's Theorem we obtain the following

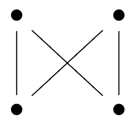
Corollary 4.2. *Let X be a finite T_0-space. Let $\mathcal{N}(\mathcal{V})$ denote the nerve of the minimal closed sets of X and denote by $\mathcal{N}(\mathcal{U})$ the nerve of its the minimal open sets. Then*

$$|\mathcal{N}(\mathcal{V})| \simeq |\mathcal{N}(\mathcal{U})|.$$

Proof. The poset $(X, \leq)$ can be identified with the poset $(\mathcal{U}, \subseteq)$ of minimal open sets of X (ordered by inclusion), each element x is identified with its minimal open set U_x. Analogously, X can be identified with the poset $(\mathcal{V}, \supseteq)$ of minimal closed sets of X. Under these identifications, the simplicial complexes K_X and L_X associated to the relation $\leq$ defined on $X \times X$, are exactly the nerves $\mathcal{N}(\mathcal{V})$ and $\mathcal{N}(\mathcal{U})$ of the minimal closed and open sets respectively. Now the result follows from the fact that $|K_X| \simeq |L_X|$. □

In the rest of the paper we investigate the simplicial complexes K_X and L_X associated to the relation $\leq$ of a finite poset $(X, \leq)$. By definition it is clear that Δ_X is a subcomplex of K_X and L_X but in general it does not have the same homotopy type as K_X (and L_X), as the following example shows.

Example 4.3. Let X be the poset with Hasse diagram



Clearly Δ_X is a one dimensional sphere S^1 but K_X and L_X are contractible.

In many situations, Δ_X is homotopy equivalent to K_X and L_X, moreover the inclusions $\Delta_X \rightarrowtail K_X$ and $\Delta_X \rightarrowtail L_X$ are deformation retracts. This follows from the following result of McCord [13].

Lemma 4.4 (McCord)**.** *Let $\mathcal{U}$ be a cover of a space with the property that the intersection of any finite collection of elements of $\mathcal{U}$ is either empty or a member of $\mathcal{U}$. Let $C(\mathcal{U})$ be the subcomplex of the nerve $\mathcal{N}(\mathcal{U})$ whose simplices are the nonempty chains of $\mathcal{U}$. Then the inclusion $C(\mathcal{U}) \rightarrowtail \mathcal{N}(\mathcal{U})$ is a deformation retract.*

If we take $\mathcal{U}$ as the covering of minimal open sets of X and use the identifications of above, we obtain

Corollary 4.5. *Let X be a finite T_0-space such that for any $x, y \in X$ the intersection $U_x \cap U_y$ is either empty or equals U_z for some $z \in X$. Then the inclusion $\Delta_X \subset L_X$ is a deformation retract.*

The posets of the form $X = \mathcal{L} - \{\hat{0}, \hat{1}\}$, with $\mathcal{L}$ a finite lattice, satisfy the hypothesis of the previous corollary. Therefore, if X is such a poset, the simplicial complexes K_X and L_X have the same homotopy type as the standard simplicial complex Δ_X. Note that Corollary 4.5 follows also from the Crosscut Theorem (see for example [6, Thm. 10.8]) and Example-Application 3.12.

4.1. **Classification of the poset structures on a finite set in terms of the K-complexes.** We investigate now how the constructions of above are related to the classification of all the poset structures that can be defined on a finite set X. More explicitly, given a subcomplex T of the simplicial complex K_* spanned by the set X, we ask whether it is possible to define a poset structure $\leq$ on X such that T is the K-complex (resp. the L-complex) of this relation.

In order to be the K-complex of a poset structure $\leq$ on X, the set of vertices of T should be the whole set X since the relation $\leq$ must be reflexive. The second condition that T must satisfy is deduced from the following lemma.

Lemma 4.6. *Let $(X, \leq)$ be a finite poset and let K_X be the associated K-complex. Let σ be a maximal simplex of K_X. Then $\sigma = U_y$ the minimal open set of y, for some maximal element $y \in X$. In particular, any maximal simplex σ of K_X contains exactly one maximal element of X and any maximal element of X is in only one maximal simplex of K_X.*

Proof. Let $\sigma = \{z_0, \ldots, z_m\}$ be a maximal simplex of K_X. Since $\sigma \in K_X$, then there exists some $y \in X$ such that $z_i \leq y$ for all i. Since σ is maximal, $y = z_j$ for some j and y must be also a maximal element of X, for if $y < w$ for some w, then we could add w to σ and this contradicts the maximality of σ. Therefore one (and only one) of the $z_i's$ is a maximal element y of X and the others are the elements of X which are smaller than y. Note that since σ is maximal, all elements smaller than y must belong to σ. □

Thus, in order to be the K-complex of a poset structure, in any maximal simplex σ of T there must be an element y which does not belong to any other maximal simplex of T. For example, the boundary of the closed 2-simplex is a simplicial complex of 3 elements which is not the K-complex of any poset structure. This is because the maximal simplices are $\{a, b\}, \{a, c\}, \{b, c\}$ (and all vertices belong to more than one maximal simplex). In general, for the same reasons, the boundary of any closed n-simplex is not a K-complex of any poset structure.

In fact, the condition of above is essentially the obstruction to be a K-complex:

Theorem 4.7. *Let T be a finite simplicial complex with vertex set X. Then T is the K-complex associated to a poset structure $\leq$ on X if and only if for any maximal simplex σ of T there exists some $y \in \sigma$ such that $y \notin \sigma'$ for all maximal simplices $\sigma' \neq \sigma$.*

Proof. One implication follows immediately from lemma 4.6 and previous remarks. To prove the other implication: Suppose T satisfies the condition on its maximal simplices. We define a poset structure on X (of length 2) as follows. Let $\sigma_1, \ldots, \sigma_r$ be the maximal simplices of T. By hypothesis, for each $i = 1, \ldots, r$ we can choose some $y_i \in \sigma_i$ such that $y_i \notin \sigma_j$ for all $j \neq i$. These $y_i's$ will be the maximal elements of the poset. Define the relation $x \leq y$ if $x = y$ or $y = y_i$ for some i and $x \in \sigma_i$. It is not difficult to prove that this is a well defined poset structure on X and that T is the K-complex of this stucture. This completes the proof. □

Remark 4.8. Note that the poset structure constructed in the proof of the Theorem is of length 2. Therefore the K-complex of any poset structure on a finite set X coincides with the K-complex of one of length 2. This implies of course that many poset structures on X have the same associated K-complex and also that for some poset structures, the associated K-complex does not have the homotopy type of the standard polyhedron Δ_X, since the complex Δ_X of a poset of length 2 is a graph (=simplicial complex of dimension 1) and any graph has the homotopy type of a bouquet of circles $\bigvee_\alpha S^1$.

From Theorem 4.7, one can also deduce:

Corollary 4.9. *K_* is the K-complex of $(X, \leq)$ if and only if $(X, \leq)$ has a maximum.*

Since the L-complex of a finite poset $(X, \leq)$ is the K-complex of the opposite (or dual) poset of $(X, \leq)$, the same result holds for the associated L-complexes.

Theorem 4.10. *Let T be a finite simplicial complex with vertex set X. Then T is the L-complex associated to a poset structure $\leq$ on X if and only if for any maximal simplex σ of T there exists some $y \in \sigma$ such that $y \notin \sigma'$ for all maximal simplices $\sigma' \neq \sigma$.*

4.2. **The complexes associated to the relation $<$.** Now we compare the K- and L-complexes associated to the relation $\leq$ with the analogous and smaller complexes associated to the relation $<$.

Let $(X, \leq)$ be a finite poset. As before, we denote by K_X and L_X the complexes induced by the relation $\leq$ and let K'_X and L'_X be the $K-$ and L-complexes associated to the relation $<$. It is clear that K'_X and L'_X are empty if the poset is discrete, so let us suppose that $(X, \leq)$ is not discrete. Moreover, we will assume that no connected component of X is a single point. By definition, the simplices of K'_X are the subsets of X consisting of elements $x_0, \ldots, x_n$ such that there is some $y \in X$ with $x_i < y$ for all i. Clearly $K'_X < K_X$. Similarly, $L'_X < L_X$. We will show that all of them K_X, K'_X, L_X, L'_X have the same homotopy type. Moreover, we will see that K_X simplicially collapses to K'_X. Similarly one can prove that L_X collapses to L'_X.

The notion of simplicial collapse was introduced by J.H.C. Whitehead in the late thirties and now it is a fundamental tool in algebraic topology and in combinatorial geometry. We recall the basic definitions.

Definition 4.11. There is an elementary simplicial collapse from a finite simplicial complex T to a subcomplex $M < T$ if there is a simplex σ of T which is contained properly in only one simplex σ' and $M = T - \{\sigma, \sigma'\}$. A finite simplicial complex K collapses to a subcomplex K' if there is a sequence of elementary collapses from K to K'. We denote in this case $K \searrow K'$.

It is easy to see that if $T \searrow M$ then $|M| \subset |T|$ is a strong deformation retract. A polyhedron is called *collapsible* if it admits a triangulation K which collapses to a point. It is clear that a collapsible polyhedron is contractible but the converse is not true. A classical example of a contractible and non collapsible polyhedron is the dunce hat (see [25]). For a comprehensive exposition on collapses and simple homotopy theory, the classical references are Whitehead's original papers [22, 23, 24]. The standard references for simple homotopy theory of CW-complexes include Milnor's article [14] and M.M.Cohen's book [7] and for the infinite case, Siebenmann's paper [18].

There is also a notion of collapse for posets (or equivalently, for finite T_0-spaces), which was introduced in a joint paper with J.A. Barmak [5]. An elementary collapse in this setting consists of removing just a single point of the poset (which is called a *weak point).* Via the associated simplicial complex Δ_X, this notion corresponds exactly to the classical notion of simplicial collapse.

We will need the following basic lemma from [22].

Lemma 4.12 (Whitehead [22])**.** *Let M, N, T be finite simplicial complexes such that $M \cap N \subset T$ and $N \searrow T$. Then $M \cup N \searrow M \cup T$.*

Now we can compare the complexes associated to the relations $\leq$ and $<$.

Theorem 4.13. *Let $(X, \leq)$ be a finite poset such that no connected component is a single point. Let K_X, L_X be the simplicial complexes associated to $\leq$ and K'_X, L'_X the complexes associated to $<$. Then $K_X \searrow K'_X$ and $L_X \searrow L'_X$.*

Proof. We prove the case $K_X \searrow K'_X$, the other case is similar.

The simplices of $K_X - K'_X$ are exactly the simplices of K_X containing some maximal element y of X. By lemma 4.6, the maximal simplices of $K_X - K'_X$ are the simplices of the form $\sigma = \{y, x_0, \ldots, x_n\}$ where y is a maximal element in X and the set $\{x_0, \ldots, x_n\}$

consists of all elements of X such that $x_i < y$. Note that the set of elements which are smaller than y is not empty by the hypothesis on the connected components of X. Since all the faces of the maximal simplex $\sigma = \{y, x_0, \ldots, x_n\}$ which contain the vertex y are not faces of any other maximal simplex σ', we can suppose without loss of generality, that $K_X - K'_X$ contains only one maximal simplex σ.

Let N be the closed $(n+1)$-simplex spanned by $\sigma = \{y, x_0, \ldots, x_n\}$ and let T be the closed n-simplex spanned by $\{x_0, \ldots, x_n\}$. Since $N \searrow T$ ([22, Lemma2]), by the lemma of above we have

$$K_X = K'_X \cup N \searrow K'_X \cup T = K'_X$$

$\square$

5. The K-complexes of closed relations

We finish the paper with a result that relates the K-complexes of two posets X, Y with *closed* relations $R \subset X \times Y$.

Definition 5.1. Given two posets $(X, \leq), (Y, \leq)$, a relation $R \subset X \times Y$ between the underlying sets is called closed if it satisfies the following property. For any $(x, y) \leq (x', y') \in X \times Y$, if $(x, y) \in R$ then $(x', y') \in R$

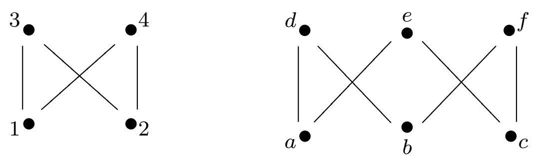
Remark 5.2. Note that, in the finite case, this is equivalent to R being a closed subset of the finite space $X \times Y$ with the product topology.

There is a well-known result by Quillen [16] (see also [6, Thm. 10.10]) which relates the closed relations with the order complexes Δ_X and Δ_Y. Explicitly,

Theorem 5.3 (Quillen). *Let X and Y be posets and let $R \subset X \times Y$ be a closed relation. For any $x \in X$ and $y \in Y$, let Δ_x be the simplicial complex of nonempty finite chains of the subposet $S_x = \{y \in Y,\ xRy\} \subset Y$ and Δ_y the corresponding simplicial complex of the subposet $S_y = \{x \in X,\ xRy\} \subset X$. If Δ_x and Δ_y are contractible for all x and y, then Δ_X and Δ_Y are homotopy equivalent.*

We will prove a variant of Quillen's result for the K-complexes. Unfortunately the analogous result (replacing $\Delta_X, \Delta_Y, \Delta_x$ and Δ_y by the corresponding K-complexes) is not valid, as we show in the following example.

Example 5.4. Consider the following posets X and Y



and the closed relation

$$R = \{(1, d), (2, e), (3, b), (3, c), (3, d), (3, e), (3, f), (4, a), (4, d), (4, e)\}.$$

It is easy to see that the K-complex of each S_x and S_y is contractible but the K-complexes of X and Y are not homotopy equivalent. The first one is contractible and the second one is homotopy equivalent to S^1. This is also an example of two posets with $|\Delta_X| \simeq |\Delta_Y|$ but with K-complexes of different homotopy types.

However the K-complexes satisfy the following weaker version of the theorem.

Theorem 5.5. *Let X and Y be posets and let $R \subset X \times Y$ be a closed relation. If the subposets S_x and S_y have maximum for all $x \in X$ and $y \in Y$, then the K-complexes K_X and K_Y are homotopy equivalent.*

Proof. Consider R as a subposet of the product poset $X \times Y$ and denote by K_R the K-complex of R. We will prove that the projection $R \to X$ (resp. $R \to Y$) induces a homotopy equivalence $K_R \to K_X$ (resp. $K_R \to K_Y$). By Quillen's Theorem A [15], it suffices to prove that $p^{-1}(\sigma) \subset K_R$ is contractible for every closed simplex σ of K_X. Let $\sigma = \{x_0, \dots, x_n\}$ be a simplex of K_X, then there exists $x' \in X$ such that $x_i \leq x'$ for all i. Let (x, y) be a vertex of $p^{-1}(\sigma) \subset K_R$. Therefore $x = x_i$ for some $i = 0, \dots, n$ and xRy. Since the relation is closed, we have that $x'Ry$. Denote by y' the maximum element of $S_{x'}$. Then $(x, y) \leq (x', y')$. We have proved that all the vertices of $p^{-1}(\sigma) \subset K_R$ are smaller than (x', y'). By definition of the K-complex, this implies that $p^{-1}(\sigma) \subset K_R$ is a closed simplex (or a generalized simplex in the infinite case) and therefore contractible. □

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